Chapter 10

Control Charts for Attributes

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10.2 Basics of p chart (Fraction Non-conforming)

A common practice is to use "percentage or fraction non-conforming" chart.

However, "fraction conforming" chart will also carry the same message.

This is known as *control chart on process yield*.

Discussion on p chart is based on binomial assumption, assuming that the probability of rejecting items is independent.

Let's assume:

- Sample size *n* (units) is selected,
- No. of defects *d* units – number of non-conforming units
- Fraction non-conforming *p*, then (*1-p*) denotes fraction conforming.

If number of non-conforming units $d = x$, then standard binomial equation applies as:

$$p(d = x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ where } x = 0, 1, 2, 3 \ldots n \qquad \ldots\ldots\ldots\ldots(1)$$

For a binomial distribution,

Mean = p

Standard deviation = $\sqrt{\dfrac{p(1-p)}{n}}$

The central limit theorem states that –

$\overline{\overline{X}} \approx \mu$ ...where, $\overline{\overline{X}}$ is the "mean of sample means" and μ is the population mean.

But, the standard deviation of the sampling distribution of sample means is –

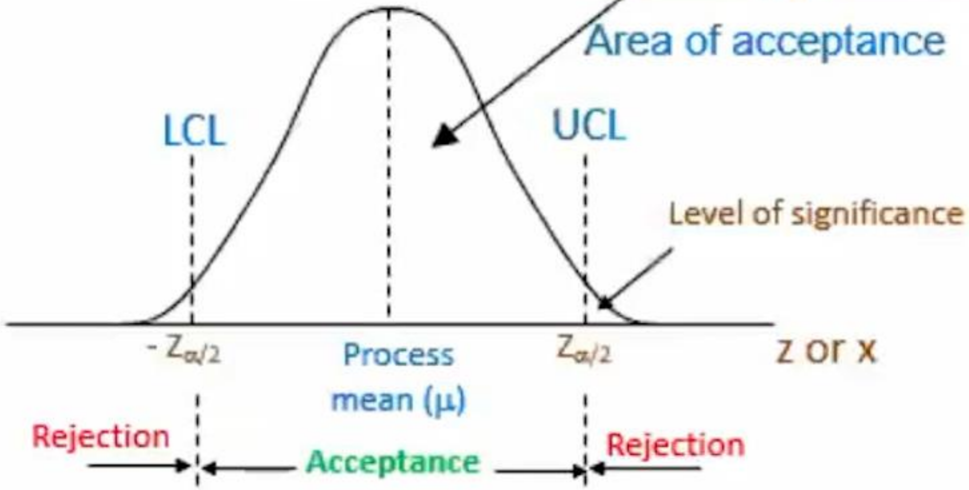
$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

[**Note:** statistical distribution of $\overline{X}$ values is called "Sampling distribution of sample means", which is approximately a Normal Distribution.]

For a two-tail test:

$$P = \mu - Z_{\alpha/2}\sigma_{\overline{X}} \leq \overline{X} \leq \mu + Z_{\alpha/2}\sigma_{\overline{X}}) = \boxed{1-\alpha}$$

where α is the level of significance and $(1-\alpha)$ is the level of confidence



$$\left.\begin{aligned} UCL &= \mu + Z_{\alpha/2}\sigma_{\bar{X}} = \mu + \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} = \bar{\bar{X}} + \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} \\ CL &= \mu = \bar{\bar{X}} \\ LCL &= \mu - Z_{\alpha/2}\sigma_{\bar{X}} = \mu - \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} = \bar{\bar{X}} - \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} \end{aligned}\right\} \quad \text{.....Eq. (3)}$$

Often, population s.d. σ is not known. In that case, an estimate of σ is taken as $\hat{\sigma}$ which is basically mean value or "Expected value" of σ –

$$\hat{\sigma} = E(\sigma) = \frac{\bar{R}}{d_2}$$

.....Eq. (6)....value of d_2 parameter can be found from Table B (p.330)

Through some computations and assumptions, we get (from Eq. 3) limits of $\bar{X}$ chart –

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2\bar{R} \qquad CL_{\bar{X}} = \bar{\bar{X}} \qquad LCL_{\bar{X}} = \bar{\bar{X}} - A_2\bar{R}$$

.........Eq. (8)

Similarly, we compute the limits of *R* chart –

$$UCL_R = \bar{R}D_4 \qquad CL_R = \bar{R} \qquad LCL_R = \bar{R}D_3$$

.........Eq. (10)

It is important to mention that the *R* chart is developed first. If the process is found in-control, only then $\bar{X}$ chart should be constructed and analyzed.

Example 1: $\overline{X} - R$ *Chart*

Metlab Casting Company Ltd. produces steel pipes of a certain diameter, considered as a critical quality characteristic. The company decided to use $\overline{X} - R$ chart to control diameter.

From a day's production, a sample of 5 pipes is selected randomly from the production line and their diameters are recorded. The average diameter and range of this sample (of size 5) are computed and recorded in a table (Table 11.1). The inspector collected this type of samples in 22 working days in the month of February. Thus $\overline{X}$ and R values of 22 samples are recorded in the table. The next step for the company is to develop trial control limits.

Day	$\overline{X}$	R	Day	$\overline{X}$	R
1	10.724	0.040	12	10.730	0.026
2	10.730	0.016	13	10.735	0.028
3	10.718	0.040	14	10.726	0.041
4	10.728	0.014	15	10.724	0.025
5	10.730	0.027	16	10.720	0.017
6	10.720	0.020	17	10.727	0.035
7	10.720	0.038	18	10.720	0.037
8	10.711	0.026	19	10.726	0.030
9	10.713	0.027	20	10.724	0.012
10	10.718	0.008	21	10.718	0.030
11	10.717	0.039	22	10.722	0.012
			Total:	235.901	0.608

Table 11.1: $\overline{X}$ and R Values for steel pipe diameters (in centimeters).

$$\overline{R} = \frac{\sum_{i=1}^{m} R_i}{m} = \frac{0.608}{22} = 0.028$$

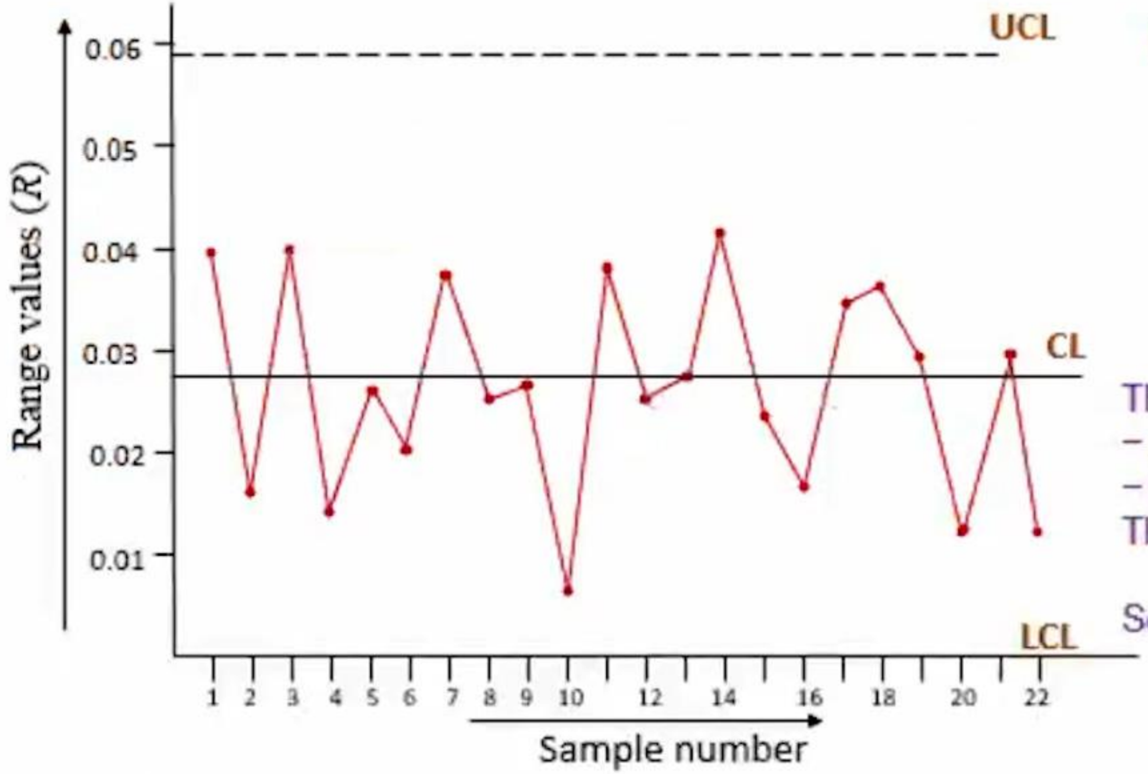
For samples of size $n = 5$,
Table B provides : $D_4 = 2.114$ and $D_3 = 0$.

From equation (10), the control limits for Range (R) chart are found –

$$UCL_R = \overline{R} D_4 = (0.028)(2.114) = 0.059$$
$$CL_R = \overline{R} = 0.028$$
$$LCL_R = \overline{R} D_3 = (0.028)(0) = 0$$

An R chart is constructed using these limits and R values (from Table 11.1) are plotted, as shown in Figure 11.2 (next slide).

Figure 11.2: *R* chart



The plots are –
– pretty random around the mean, and
– within the limits.
Thus, in-control.

So, $\overline{X}$ chart can be plotted now.

$$\overline{\overline{X}} = \frac{\sum_{i=1}^{22} \overline{X}_i}{22} = \frac{235.901}{22} = 10.722$$

For samples of size n = 5, Table B provides : $A_2 = 0.577$,

$$\text{UCL}_{\overline{X}} = \overline{\overline{X}} + A_2\overline{R} = 10.722 + 0.577(0.02\ 8) = 10.738$$
$$\text{CL}_{\overline{X}} = 10.722$$
$$\text{LCL}_{\overline{X}} = \overline{\overline{X}} - A_2\overline{R} = 10.722 - 0.577(0.028) = 10.706$$

An $\overline{X}$ chart is constructed using these limits, and $\overline{X}$ values (from Table 11.1) are plotted, as shown in Figure 11.3 (next slide).

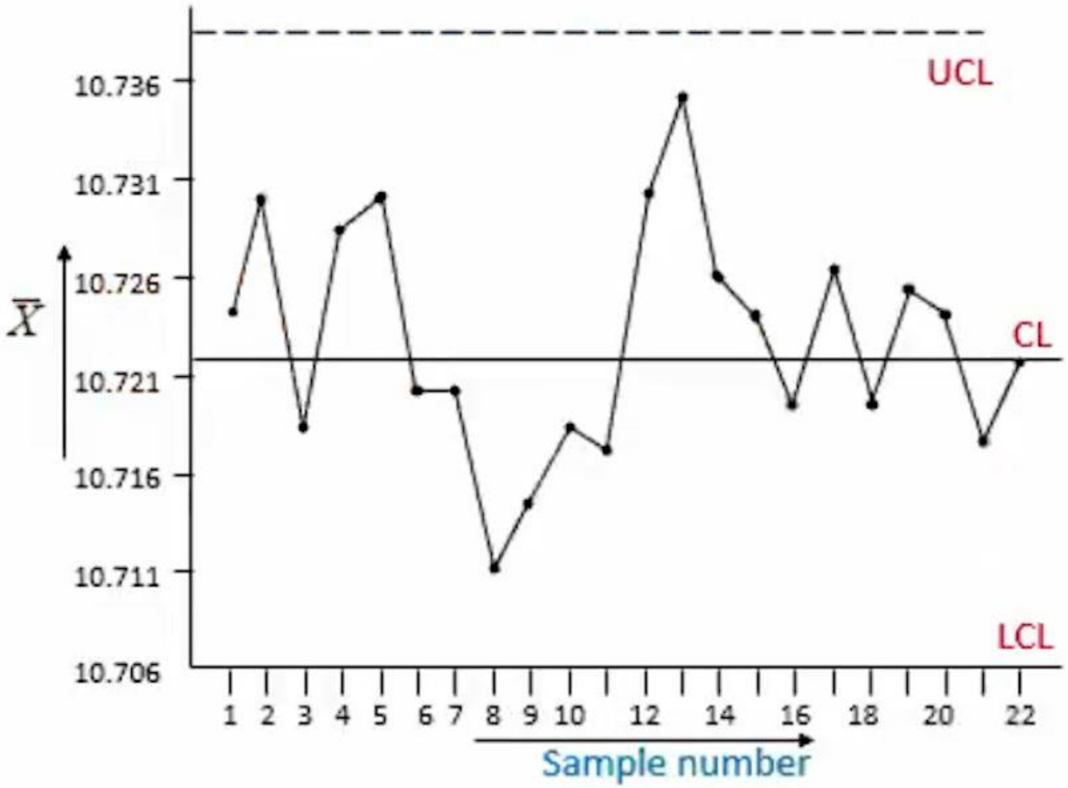


Figure 11.3: Control chart.

The plots are –

- pretty random around the mean, and
- within the limits.

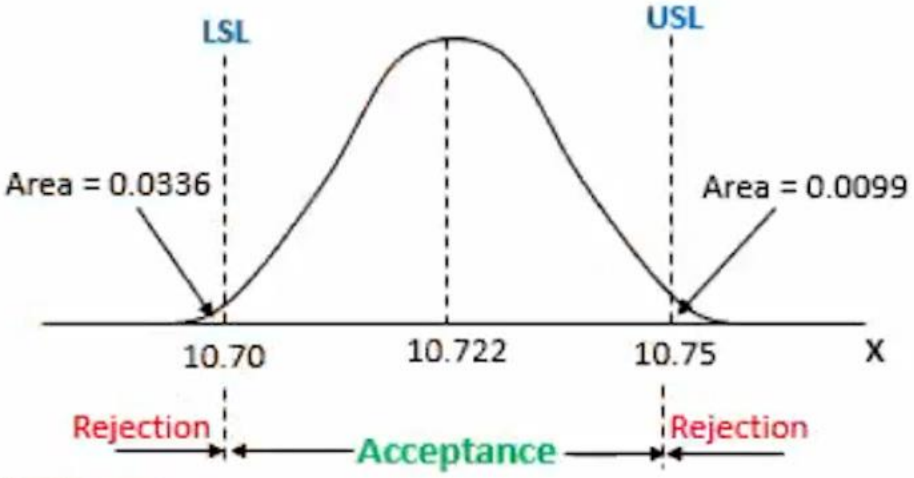
Thus, in-control.

So, we conclude that the process is fairly in-control.

Process Capability Analysis

$$\hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{0.028}{2.326} = 0.012$$

Suppose that the stated speciation limits are : 10.70 cm, 10.75 cm, as illustrated in the Figure (Figure 11.4).



Fraction non-conforming at the right and left tails are:

$$P(X > USL) = P\left(Z > \left[\frac{10.75 - 10.722}{0.012}\right]\right) = P(Z > 2.33) = 0.0099 = 0.99\%$$

$$P(X < LSL) = P\left(Z < \left[\frac{10.70 - 10.722}{0.012}\right]\right) = P(Z < -1.83) = 0.0336 = 3.36\%$$

Total fraction of 'products not meeting specifications' is : 0.0099 + 0.0336 = 0.0435 or 4.35%, which is quite high for a good process.

Process capability ratio, where σ is estimated by $\hat{\sigma}$, is –

$$PCR = \frac{USL - LSL}{6\hat{\sigma}} = \frac{10.75 - 10.70}{6 \times 0.012} = 0.694 < 1$$ ……**Process Potential Index C_p**

PCR value is alarmingly less than 1, which indicates that a large number of products will be nonconforming.

One important point to notice that proportions of products not meeting specifications on two sides (or the tails) of the normal distribution are not equal which further reveals marginal shifting of the process mean.

11.3 $\bar{X}$–S chart

Range (R) is not a good measure of variability, although it is easy to compute, as far as computational complexity is concerned.

Standard deviation is a better measure in all respects.

Thus, $\bar{X}$–S chart is better than $\bar{X}$–R chart, where S is sample standard deviation.

When number of data elements, i.e. the sample size n, increases further, say $n > 10$, range chart looses its credibility completely.

However, S chart is suggested, because in majority of the cases, population standard deviation σ is unknown.

11.3.1 The concepts of $\overline{X} - S$ chart

Sample standard deviation is -

$$S_i = \sqrt{\frac{\sum_{j=1}^{n} (x_j - \bar{x})^2}{n-1}}$$ .....for Sample of size n

Mean value of sample s.d. of *m* samples is – $\overline{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$

The control limits for $\overline{X}$ chart are –

$$\left.\begin{aligned} UCL_{\overline{X}} &= \overline{\overline{X}} + A_3\overline{S} \\ CL_{\overline{X}} &= \overline{\overline{X}} \\ LCL_{\overline{X}} &= \overline{\overline{X}} - A_3\overline{S} \end{aligned}\right\}$$ ....Eq. (11)

The control limits for S chart are –

$$\left.\begin{aligned} UCL_S &= B_4\overline{S} \\ CL_S &= \overline{S} \\ LCL_S &= B_3\overline{S} \end{aligned}\right\}$$ ....Eq. (12)

Interpretation and analysis of this chart is also similar to $\overline{X}$- R chart.

$$UCL = \bar{c} + 3\sqrt{\bar{c}} = 12.36 + 3\sqrt{12.36} = 22.90$$

$$CL = \bar{c} = 12.36$$ …… these are the first trial limits

$$LCL = \bar{c} - 3\sqrt{\bar{c}} = 12.36 - 3\sqrt{12.36} = 1.81$$

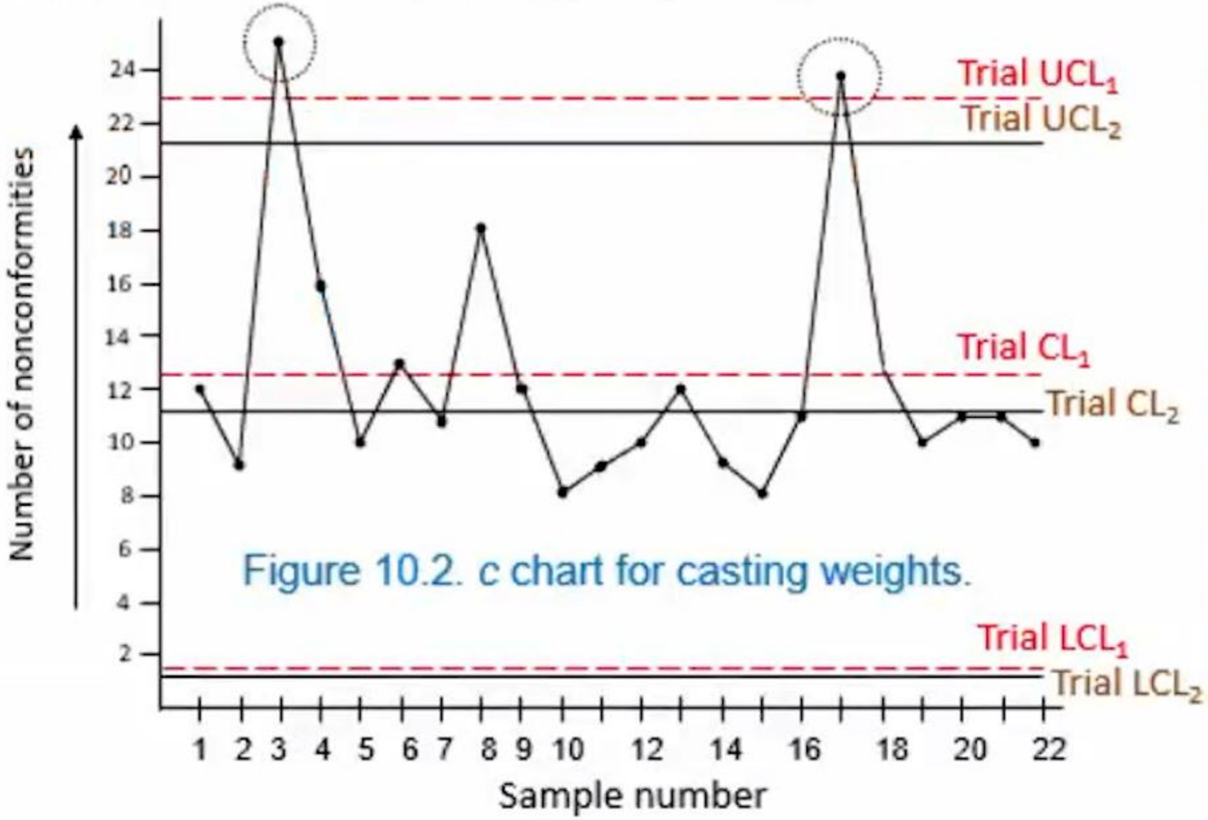


Figure 10.2. *c* chart for casting weights.

Observations:

- The *c* control chart indicate fair degree of randomness, but just below the mean (CL).
- Two samples (number 3 and 17) falling outside of the upper control limits.

Investigations into the problems revealed assignable causes.

On day 3 (sample 3), pressure of gas supply in the national grid was very low, thereby creating lower than required heat in the furnace. On day 17, materials from a new supplier, instead of a regular supplier, was used. The regular supplier had strike in their organization. Thus, these two values are omitted. The second trial limits are calculated as follows in the next slide.

For the revised 2nd trial, mean no. of nonconformities in each inspection unit is –

$$\bar{c} = \frac{223}{20} = 11.15$$

The revised 2nd trial limits are given below, which are plotted in the same Figure 10.2:

$$\text{UCL} = \bar{c} + 3\sqrt{\bar{c}} = 11.15 + 3\sqrt{11.15} = 21.16$$

$$\text{CL} = \bar{c} = 11.15$$

$$\text{LCL} = \bar{c} - 3\sqrt{\bar{c}} = 11.15 - 3\sqrt{11.15} = 1.13$$

Decision:

The new limits demonstrate that none of the 20 plots are out of control.

The plots are also more random around the mean or center line.

Thus, these limits can be considered as the final control limits.

Example: Type I and II Errors in c Chart (First trial)

Mean = $\bar{x} = \bar{c} = 12.36$; UCL = 22.90; LCL = 1.81

Standard Deviation $\sigma = \sqrt{Var} = \sqrt{\bar{c}} = \sqrt{12.36} = 3.52$

It is to be noted that number of nonconformities is **discrete** in nature (Poisson Distribution), whereas the following discussion is based on Normal (**continuous** distribution) assumption / approximation.

[Type I error : a sample value falls outside of the control limits when the process is in control].

Probability of Type I error (α) = $P(X < 1.81) + P(X > 22.90) = P(X \leq 2) + P(X \geq 23)$

The assumption of normal approximation can be used as below:

$$P(X < 2) = P\left[Z < \left\{\frac{(2+0.5)-12.36}{3.52}\right\}\right] = P(Z < -2.8) = 0.5 - 0.4974 = 0.0026$$

$$P(X > 23) = P\left[Z > \left\{\frac{(23-0.5)-12.36}{3.52}\right\}\right] = P(Z > 2.88) = 0.5 - 0.4980 = 0.0020$$

Thus, $\alpha = 0.0026 + 0.0020 = 0.0046$

Two important points to notice that –

1) This probability value (α) is greater than the standard value of 0.0027, as Empirical rule says, in 3σ control limits,
2) Two individual probabilities are not equal on both sides of the tails, which indicates that the Poisson distribution is asymmetric on two tails.

[Type II error: a sample value falls inside the control limits when the process is out-of-control].

Probability of Type II error (β) = P(X < 22.90) – P(X < 1.81)
= P (X ≤ 23) – P(X ≤ 2)

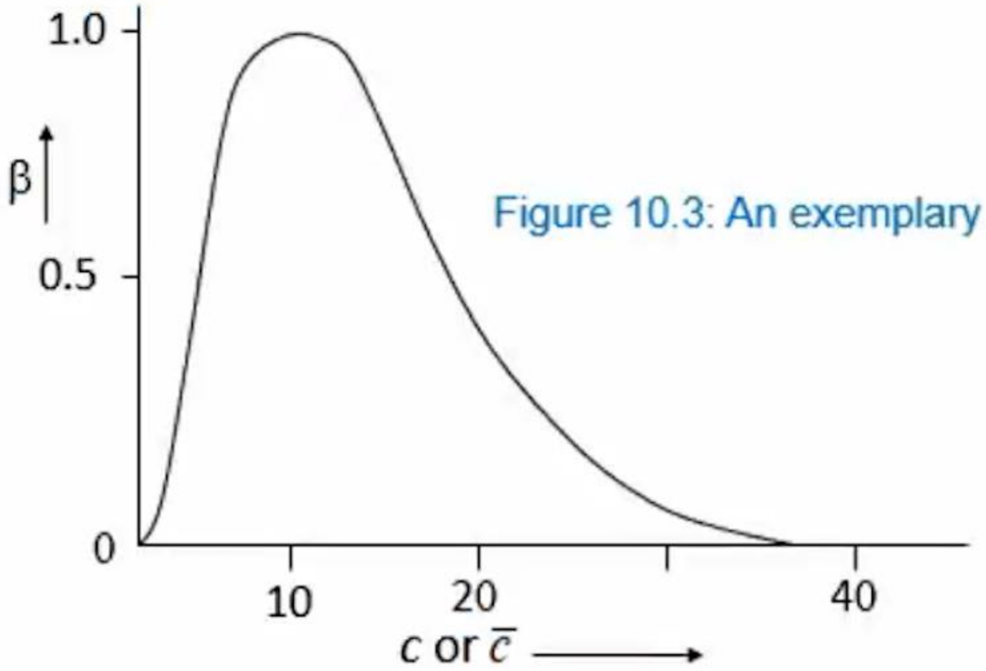


Figure 10.3: An exemplary OC curve for Type II error.

10.5 The u Chart

In *c* chart, a fixed size of 'one inspection unit" is considered (for instance, the example considered 30 pieces in one inspection unit).

However, in many cases, the size of inspection unit may vary time to time.

For instance, the inspection unit may be of different sizes, such as 20 pieces, 25 pieces, 32 pieces, etc. on different days.

In such a case, "number of nonconformities per unit" (u chart) can be used.

Another instance:

Say, number of blemishes on a 50 m^2 wall and that on a 500 m^2 wall, may also used u chart.

Such type of situations can also be handled using variable control limits for different samples (see p-179).

We will not study this in detail.